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Problems of Nonextensivity in Hadron Thermodynamics

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Hadron thermodynamics deals with a gas of indistinguishable particles whose mass spectrum is taken to have the form $Km^a e^{bm}$. A long-standing inconsistency is pointed out, namely the nonextensive nature of the entropy found in some treatments by way of the microcanonical ensemble, which contrasts with the extensive nature found by way of the canonical ensemble. The former result is due to an error. After correction, the two ensembles are found to lead to the same expressions for the thermodynamic quantities if $a \geqslant -7/2$. Some of these expressions are new. For $a < -5/2$, the microcanonical approach is used to examine a model in which one particle is appreciably heavier than the rest. However, the resulting entropy is found to be unphysical.

KEY WORDS: Entropy; nonextensivity; thermodynamics; hadrons.

1. INTRODUCTION

In hadron thermodynamics, the density of hadron masses is given by

$$\rho(m) = Km^a e^{bm} \tag{1.1}$$

where K, a, and b are constants.[1] We are interested in whether or not the resulting entropy is extensive (i.e., scales with the size of the system). Many systems in thermodynamics are entensive in that sense. Those with long-range interactions are not, the black hole case being a significant example. However, even such systems retain the property of superadditivity for their entropies. For example, consider two adiabatically isolated systems A and B which can be brought into thermal contact by the withdrawal of a par-

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tition. Since this is an adiabatic process, the entropy cannot decrease as a result of this procedure, so that

$$S_{AB} \geqslant S_A + S_B \tag{1.2}$$

It can be seen that (1.2) encapsulates the essence of the second law even for nonextensive systems,[2] there being no need to appeal to other principles [such as the subadditivity of the energy $E(S_A + S_B) \leqslant E(S_A) + E(S_B)$]. One would expect that a change in the value of a in (1.1) would not alter as basic a property as extensivity or supperadditivity. This view leads one to track down some discrepancies in hadron thermodynamics, whose age in no way detracts from their intrinsic interest. Unless they are cleared up, the whole theory is in doubt once the discrepancies have been pointed out. The aim of this paper is to save the theory thus threatened by removing the discrepancies, which are listed in Table I, where σ denotes the phase space density for a system of particles described by a level density (1.1). This applies to an approach via the microcanonical ensemble.

The original approach via the canonical ensemble[1] showed no discrepancies and the entropy was found to be extensive, as confirmed in Ref. 4 and in the present paper.

We now explain the origin of the discrepancies. They are purely mathematical and are due to the unnecessary introduction of the quantity $\pi^{(n)}(E)$ in Eq. (3.14) of Ref. 4, subsequent estimates of which are really those for the n-particle contribution $\sigma^{(n)}$ to the phase space density. We show here that once $\sigma^{(n)}$ is estimated correctly, the extensive nature of the entropy emerges for the microcanonical treatment also. As might be expec-

Table I. Properties of the Entropy of a Hadron Gas Derived via the Microcanonical Approach for Various Values of the Parameter *a*

Value of a	Nature of the entropy as given in the original paper orinferred by us from $S = k \ln \sigma$	Ref.
$a > -5/2$	Extensive	3
	Nonextensive	4
$a = -5/2$	Nonextensive	3, 4
$-7/2 < a < -5/2$	Nonextensive	4
$a = -7/2$	Nonextensive	4
$a < -5/2^{a}$	Nonextensive but superadditive	3
	Nonextensive but not superadditive	4

[a] Special case when one particle is singled out to be appreciably heavier than the rest.

ted, we find new corrected expressions for the thermodynamic quantities as part of the recalculation given in Section 3 and this applies to the first four of the five cases treated there. This section contains the main part of this paper.

This still leaves the question of the grand canonical treatment. Since particle creation is essential, only the case of zero chemical potential need be considered. Hence, one can show [Eq. (2.2)] that in the thermodynamic limit the canonical partition function and the grand canonical partition function are the same. This makes consistency between these two ensembles automatic for our case. The rest of Section 2 contains the derivation of the other results for the canonical/grand canonical ensemble.

Thus, the present paper converts a theory flawed by an inconsistency into a tenable one. In particular, there are no negative heat capacities. It is their suggested existence[4] that first gave rise to this investigation. They occur only for systems with nonextensive entropies.[2] We have shifted the main emphasis of the analysis from the phase space densities to entropies. The latter are considered in Ref. 4 only in Eq. (4.12).

In conclusion, we note two additional points:

1. A special case for $a < -5/2$ when one particle is singled out to be appreciably heavier than the rest is also discussed.[3,4] In both cases the resulting entropy expression is nonextensive. However, while one is found to be superadditive, thus making it physically acceptable,[3] the other is not superadditive[4] and so must be unphysical. It is shown here in case (v) of Section 3 that using the corrected calculation of the phase space density still results in an entropy that is nonextensive and not superadditive. Therefore, the model considered is unphysical.

2. In agreement with earlier work (see, for example, Ref. 5), the partition function converges at $T = T_0$ only if $a < -5/2$. The energy of the system converges at $T = T_0$ only if $a < -7/2$. Hence, for $a \geqslant -7/2$, the temperature T_0 is attained only in the limit of infinite energy and so constitutes an ultimate temperature for the system.

The relationship between descriptions of the same system by different ensembles, which is our main concern here, arises in a very similar form in the theory of string excitations and its application to the early universe and black hole evaporation.[6] The number of dimensions used can then, of course, be greater than four, possibly 10 or 26. However, insofar as thermodynamics applies, it must be true that a superadditive entropy is consistent with a negative heat capacity if and only if the entropy is nonextensive.[2] Therefore, the topic discussed here is of current interest in string and superstring theory.

2. CANONICAL/GRAND CANONICAL APPROACHES

Consider a system of identical particles described in both the grand canonical ensemble (gce) and the canonical ensemble (ce). If V and T are its volume and absolute temperature, the Helmholtz free energies satisfy

$$F_{\text{gce}}(\mu, V, T) = \mu\bar{N} - pV = \mu\bar{N} - kT\ln \Xi(\mu, V, T)$$
$$F_{\text{ce}}(N, V, T) = -kT\ln Z(N, V, T)$$

Assume that $\bar{N}(\mu, V, T)$ in the grand canonical ensemble is the same as N in the canonical ensemble. Thus

$$V^{-1}[F_{\text{gce}}(\mu, V, T) - F_{\text{ce}}(N, V, T)] = \frac{kT}{V}[\ln Z(N, V, T) - \ln \Xi(\mu, V, T)] + \frac{\mu\bar{N}}{V} \tag{2.1}$$

The limit $\mu \to 0$ may be taken provided it is recognized that V, T must be fixed while $\bar{N}(\mu, V, T) = N$ changes appropriately. The last term in (2.1) now vanishes provided $\bar{N}(0, V, T)$ is finite. Next take the thermodynamic limit in (2.1). If the system is extensive, the two ensembles then give the same results[7] and so the left-hand side of (2.1) also vanishes. This gives

$$V^{-1}\ln Z(N, V, T) \sim V^{-1}\ln \Xi(0, V, T) \tag{2.2}$$

for extensive systems, where $\sim$ denotes equality in the thermodynamic limit. If the limits are taken in reverse order, the same result is found provided $\bar{N}/V$ is finite in the thermodynamic limit. The result (2.2) will hold also if a whole spectrum of identical particles is present, as will be assumed below. This new (or little known) result will be used here for calculating results in the grand canonical ensemble.

Thus, in the canonical ensemble (e.g., Ref. 4), but also in the grand canonical ensemble (not considered in Ref. 4), the standard result

$$\begin{aligned} V^{-1}\ln Z &\sim \frac{T}{2\pi^2}\int_{m_0}^{\infty} \rho(m) \sum_{n=1}^{\infty} \frac{m^2}{n^2}\left(\frac{\pi T}{2nm}\right)^{1/2} e^{-nm/T}\,dm \\ &= \frac{KT^{3/2}}{(2\pi)^{3/2}}\int_{m_0}^{\infty}\sum_{n=1}^{\infty}\frac{m^{a+3/2}}{n^{5/2}}\exp\left[m\left(\frac{1}{T_0}-\frac{n}{T}\right)\right]dm \end{aligned} \tag{2.3}$$

may be used. The units are such that $\hbar = c = k = 1$, the spectrum (1.1) is assumed with $b = 1/T_0$, and m_0 is the smallest rest mass considered. Carrying out the integration, we obtain

$$V^{-1}\ln Z \sim \frac{KT^{3/2}}{(2\pi)^{3/2}}\sum_{n=1}^{\infty}\frac{1}{n^{5/2}}\left(\frac{n}{T}-\frac{1}{T_0}\right)^{-a-5/2}\int_{x_{n0}}^{\infty} x^{a+3/2}e^{-x}\,dx \tag{2.4}$$

where $x_{n0} \equiv m_0(n/T - 1/T_0)$. For large but finite values, $\ln Z \propto V$, so that the pressure is

$$P = T\,\partial \ln Z/\partial V \Rightarrow P \sim TV^{-1} \ln Z \tag{2.5}$$

The well-known result (e.g., Ref. 1),

$$T \leqslant T_0 \quad \text{for all values of } a \qquad \text{provided } x_{n0} > 0 \text{ for all } n \tag{2.6}$$

implying that T_0 is a limiting temperature, is obtained by considering the $n = 1$ term in (2.3), which converges only if (2.6) holds. The limit $T = T_0$ can be reached only if $\int_{m_0}^{\infty} m^{a+3/2}\,dm$ converges; that is.

$$T = T_0 \quad \text{is possible only if } a < -5/2 \tag{2.7}$$

It is seen that $V^{-1} \ln Z$ and hence its derivatives P and the energy per unit volume E/V are all intensive. In that sense, then, E and S are extensive quantities in this model.

Note explicitly

$$\frac{E}{V} \sim \frac{3T}{2V} \ln Z + \frac{KT^{3/2}}{(2\pi)^{3/2}} \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \left(\frac{n}{T} - \frac{1}{T_0}\right)^{-a-7/2} \int_{x_{n0}}^{\infty} x^{a+5/2} e^{-x}\,dx \tag{2.8}$$

In the neighborhood of $T \sim T_0$ this is

$$\frac{E}{V} \sim \frac{K}{(2\pi)^{3/2}} \sum_{n=1}^{\infty} \frac{1}{n^{5/2}} \frac{T_0^{2a+15/2}}{(nT_0 - T)^{a+7/2}} \left[\frac{3}{2}(nT_0 - T) \int_{x_{n0}}^{\infty} x^{a+3/2} e^{-x}\,dx + nT_0 \int_{x_{n0}}^{\infty} x^{a+5/2} e^{-x}\,dx\right] \tag{2.9}$$

Observe that

results (2.8), (2.9) hold for all values of a provided $x_{n0} > 0$ (2.10a)

if $x_0 = 0$, it is required that $a < -5/2$ in order that $V^{-1} \ln Z$ be finite (2.10b)

Also note that, in the expression for E/V, the $n = 1$ term will dominate and, for values of T approaching T_0, it is seen from (2.9) that this will be given approximately by

$$\frac{K}{(2\pi)^{3/2}} \frac{T_0^{2a+17/2}}{(T_0 - T)^{a+7/2}} \int_{x_{10}}^{\infty} x^{a+5/2} e^{-x}\,dx \tag{2.11}$$

If one of the two conditions (2.10) is fulfilled, the heat capacity at constant volume C_V and E are both proportional to V for large but finite volumes. Then, the mean square relative fluctuation in the internal energy is given by

$$\frac{\overline{(E-\bar{E})^2}}{(\bar{E})^2}=\frac{kT^2C_v}{E^2}\propto\frac{1}{V}$$

and so vanishes in the thermodynamic limit. Thus, one obtains sensible results by this approach.

3. MICROCANONICAL APPROACH

The phase space density for a system of particles described by a level density (1.1) is

$$\sigma=\sum_{n=1}^{\infty}\left[\frac{V}{(2\pi)^3}\right]^n\frac{1}{n!}\prod_{i=1}^{n}\int_{m_0}^{\infty}dm_i\,Km_i^a\int d^3p_ie^{bm_i}\,\delta\left(\sum_{i=1}^{n}E_i-E\right)$$

The energy of the ith particle may be written in terms of its kinetic energy Q_i by

$$E_i=m_i+Q_i$$

so that

$$\prod_{i=1}^{n}e^{bm_i}=e^{bE}\prod_{i=1}^{n}e^{-bQ_i}$$

Also, for a nonrelativistic gas, $p_i\sim(2m_iQ_i)^{1/2}$, and so, in this case, following Ref. 4, the phase space density may be estimated by

$$\sigma\simeq\sum_{n=1}^{\infty}\left[\frac{KV}{(2\pi b)^{3/2}}\right]^n\frac{e^{bE}}{n!}\prod_{i=1}^{n}\int_{m_0}^{A_i}dm_im_i^{a+3/2}\tag{3.1}$$

where the cutoff A_i is introduced to take approximate account of energy conservation, following Refs. 3 and 4. Five special cases will be examined and corrected in the remainder of this section.

Case (i) $a>-5/2$. In this case, the mass integral in (3.1) is dominated by states with large mass and is seen to have the approximate value

$$A_i^{a+5/2}/(a+5/2)$$

The n-particle contribution to σ, denoted by $\sigma^{(n)}$, may be evaluated subject to the constraint

$$\sum_{i=1}^{n} \Lambda_i = E$$

The maximum contribution to $\sigma^{(n)}$ is seen to be obtained when the Λ_i are all of order E/n and this provides the estimate

$$\sigma^{(n)} \simeq \frac{e^{bE}}{n!}\left[\frac{KV}{(2\pi b)^{3/2}}\frac{1}{a+5/2}\left(\frac{E}{n}\right)^{a+5/2}\right]^{n} \tag{3.2}$$

Following Ref. 3, $\sigma^{(n)}$ is seen to possess a maximum when

$$n = N \simeq \left[\frac{KV}{(2\pi b)^{3/2}}\, e^{-a-5/2}\, \frac{E^{a+5/2}}{a+5/2}\right]^{1/(a+7/2)}$$

As noted in Ref. 4, for large E, N grows like $E^{(a+5/2)/(a+7/2)}$ and so the mean energy per particle and, hence, the average mass are large when the energy density is sufficiently large. This justifies the approximations used to obtain (3.2). Using the above results and estimating $\sigma \simeq \sigma^{(N)}$ leads to the following expression for the entropy:

$$S = \ln \sigma \simeq bE + \left(a+\frac{7}{2}\right)\left[\frac{KV}{(2\pi b)^{3/2}}\frac{e^{-a-5/2}E^{a+5/2}}{a+5/2}\right]^{1/(a+7/2)}$$

and this expression is seen to be extensive [contrast Ref. 4, Eq. (3.23)].

Again,

$$\frac{1}{T} = \frac{\partial S}{\partial E} \simeq b + \left(a+\frac{5}{2}\right)\left[\frac{KV}{(2\pi b)^{3/2}}\frac{e^{-a-5/2}}{(a+5/2)\,E}\right]^{1/(a+7/2)}$$

or

$$\frac{E}{V} \simeq \frac{K}{(2\pi)^{3/2}}\left(\frac{a+5/2}{e}\right)^{a+5/2} T_0^{2a+17/2}(T_0 - T)^{-a-7/2}$$

for values of T approaching T_0.

Apart from a slight difference in the coefficient, this latter expression is seen to agree with (2.11), which was obtained via the canonical/grand canonical approach, for the case $a > -5/2$.

Case (ii). $a = -5/2$. In this case, the mass integral in (3.1) is dominated once again by states with large mass and is seen to have the value

$$\ln(\Lambda_i/m_0)$$

As in case (i), $\sigma^{(n)}$ will receive its main contribution for $\Lambda_i \sim E/n$ and this leads to the estimate

$$\sigma^{(n)} \simeq \frac{e^{bE}}{n!}\left[\frac{KV}{(2\pi b)^{3/2}} \ln\left(\frac{E}{nm_0}\right)\right]^n \tag{3.3}$$

Using the approach of case (1), we find that this latter expression possesses a maximum when

$$n = N \simeq \frac{KV}{(2\pi b)^{3/2}} \ln \frac{E}{Nm_0}$$

The sum over n of $\sigma^{(n)}$ may be accomplished approximately by replacing the n in the argument of the logarithm in (3.3) by N. Since $\ln n$ varies slowly compared with $n!$, this is an acceptable approximation. Hence, the entropy is given by

$$S = \ln \sigma = \ln \sum_{n=1}^{\infty} \sigma^{(n)} \simeq bE + \frac{KV}{(2\pi b)^{3/2}} \ln \frac{E}{Nm_0}$$

and this expression is seen to be extensive [constrast Ref. 4, Eq. (3.32)].

Again,

$$\frac{1}{T} = \frac{\partial S}{\partial E} \simeq b + \frac{KV}{(2\pi b)^{3/2}} \frac{1}{E}$$

or

$$\frac{E}{V} \simeq \frac{K}{(2\pi)^{3/2}} \frac{T_0^{7/2}}{T_0 - T}$$

for values of T approaching T_0.

This result is seen to agree with (2.11) for $a = -5/2$.

Case (iii). $-7/2 < a < -5/2$. As pointed out in Ref. 4, the case when $a < -5/2$ is somewhat different from those considered previously. In this case, the mass integral in (3.1) has the value

$$\frac{m_0^{a+5/2} - \Lambda_i^{a+5/2}}{-a-5/2}$$

and so receives its main contribution from the low-mass region. Hence, important contributions may be made to $\sigma^{(n)}$ by configurations in which $n-1$ particles have small masses (with a mean mass $\bar{m}$) and one particle has a mass fixed by energy conservation to be of order $E-(n-1)\,\bar{m}$. These configurations will be dominant when E becomes sufficiently large,

$$E \gg n\bar{m}(E) \tag{3.4}$$

The mean mass in the mass integral in (3.1) is given by

$$\bar{m}(\Lambda_i) = \int_{m_0}^{\Lambda_i} m^{a+5/2}\, dm \Big/ \int_{m_0}^{\Lambda_i} m^{a+3/2}\, dm$$

from which it follows that

$$\begin{aligned} &\bar{m}(\Lambda_i) \simeq [(-a-5/2)/(a+7/2)]\, \Lambda_i^{a+7/2}/m_0^{a+5/2} && \text{for} \quad -5/2 > a > -7/2 \\ &\bar{m}(\Lambda_i) \simeq m_0 \ln(\Lambda_i/m_0) && \text{for} \quad a = -7/2 \\ &\bar{m}(\Lambda_i) \simeq [(a+5/2)/(a+7/2)]\, m_0 && \text{for} \quad a < -7/2 \end{aligned} \tag{3.5}$$

If condition (3.4) is not met, energy conservation may be imposed in the approximate form

$$\bar{m}(\Lambda_i) = E/n$$

Then, in the case when $-5/2 > a > -7/2$, Λ_i is seen to be given by

$$\Lambda_i = \left(\frac{a+7/2}{-a-5/2}\, m_0^{a+5/2} \frac{E}{n} \right)^{1/(a+7/2)} \tag{3.6}$$

and it follows that

$$\sigma^{(n)} \simeq \frac{e^{bE}}{n!} \left\{ \frac{KVm_0^{a+5/2}}{(2\pi b)^{3/2}\,(-a-5/2)} \left[1-(\Lambda_i/m_0)^{a+5/2}\right] \right\}^n \tag{3.7}$$

with Λ_i given by (3.6).

Using the approach adopted for the previous cases considered, we see that this expression for $\sigma^{(n)}$ is maximal for

$$n = N \simeq \frac{KV}{(2\pi b)^{3/2}} \frac{m_0^{a+5/2}}{(-a-5/2)} \tag{3.8}$$

Estimating $\sigma \simeq \sigma^{(N)}$ leads to the following expression for the entropy:

$$S = \ln \sigma \simeq bE + N + N \ln \left\{ \frac{KV}{(2\pi b)^{3/2}} \frac{m_0^{a+5/2}}{N(-a-5/2)} \left[1 - \left(\frac{\Lambda_i}{m_0} \right)^{a+5/2} \right] \right\} \tag{3.9}$$

where Λ_i is given by (3.6) with $n = N$.

This expression for the entropy is seen to be extensive (contrast Ref. 4). Again,

$$\frac{1}{T}=\frac{\partial S}{\partial E}\simeq b+\left[\frac{KV}{(2\pi b)^{3/2}}\frac{1}{E\,(a+7/2)}\right]^{1/(a+7/2)}$$

or

$$\frac{E}{V}\simeq\frac{K}{(2\pi)^{3/2}}\frac{1}{a+7/2}\frac{T_0^{2a+17/2}}{(T_0-T)^{a+7/2}}$$

for values of T approaching T_0.

Apart from small differences in the numerical coefficients, which are understandable considering the approximations being made, this result is seen to agree with expression (2.11) when $-5/2>a>-7/2$.

Case (iv). $a=-7/2$. Assume that condition (3.4) is still not satisfied. In this case, A_i is given by

$$A_i\simeq m_0 e^{E/nm_0} \tag{3.10}$$

and $\sigma^{(n)}$ is given by (3.7) once again, but with A_i as in (3.10). This expression for $\sigma^{(n)}$ is seen to be maximal for $n=N$ as given by (3.8) once more. Estimating $\sigma\simeq\sigma^{(N)}$ leads to (3.9) as the expression for the entropy, but with A_i as in (3.10) with $n=N$. This expression for the entropy is seen to be extensive (contrast Ref. 4). Again,

$$\frac{1}{T}=\frac{\partial S}{\partial E}\simeq b+\frac{e^{-E/Nm_0}}{m_0}$$

or

$$\frac{E}{V}\simeq\frac{K}{(2\pi)^{3/2}}T_0^{3/2}\ln\frac{T_0^2}{m_0(T_0-T)}$$

for values of T approaching T_0.

This result is seen to agree with expression (2.11) for the case when $a=-7/2$.

Hence, it has been shown that, for $a\geqslant -7/2$, the microcanonical and canonical/grand canonical descriptions are equivalent, as expected. Unfortunately, due to the form of the approximate expression (3.5) for $\bar{m}(A_i)$ in the case when $a<-7/2$, the present microcanonical approach proves unsuitable for an investigation of the model under discussion in that case.

Case (v). $a < -5/2$ with (3.4) holding. However, one further case was considered in Ref. 4 and should be mentioned here for completeness—this is the case where $a < -5/2$, but condition (3.4) holds. As mentioned earlier, the main contributions to $\sigma^{(n)}$ in this case were felt to come from configurations in which $n-1$ particles have small masses with mean mass $\bar{m}(E)$ and one particle has a mass fixed by energy conservation to be of order $E-(n-1)\,\bar{m}(E)$. From (3.1) the contribution of such configurations is seen to be estimated by

$$\sigma^{(n)} \simeq \frac{e^{bE}}{(n-1)!}\left[\frac{KVm_0^{a+5/2}}{(2\pi b)^{3/2}\,(-a-5/2)}\right]^{n-1}\frac{KV}{(2\pi b)^{3/2}}\frac{(m_0^{a+5/2}-E^{a+5/2})}{(-a-5/2)} \tag{3.11}$$

where the factor $1/(n-1)! = n/n!$ arises since the high-mass particle may be associated with any of the n mass integrals and the mass of the heavy particle has been approximated by E, using (3.4).

As before, the most probable number of particles N is found by maximizing $\sigma^{(n)}$ with respect to n, and once again the result is given by (3.8). Then, the sum of (3.11) over n may be performed to give

$$\sigma = \sum_{n=1}^{\infty} \sigma^{(n)} \simeq e^{bE}\,\frac{KV(m_0^{a+5/2}-E^{a+5/2})}{(2\pi b)^{3/2}\,(-a-5/2)}\exp\frac{KVm_0^{a+5/2}}{(2\pi b)^{3/2}\,(-a-5/2)}$$

from which it follows that the entropy is given by

$$S = \ln\sigma \simeq bE + \frac{KVm_0^{a+5/2}}{(2\pi b)^{3/2}\,(-a-5/2)} + \ln\frac{KV(m_0^{a+5/2}-E^{a+5/2})}{(2\pi b)^{3/2}\,(-a-5/2)}$$

It is immediately apparent that this entropy expression is not extensive. Also, it is seen to be a concave function, since $\partial^2 S/\partial E^2$ and $\partial^2 S/\partial V^2$ are both negative, but $\partial^2 S/\partial V\,\partial E$ is zero. Therefore, since it has been shown[2] that concavity and superadditivity together imply extensivity, it follows that the above entropy expression is not superadditive, and hence is unphysical.

4. CONCLUSIONS

The statistical thermodynamics of a hadron gas have been discussed first via the canonical/grand canonical approach. The system was found to possess an extensive entropy function and the mean square relative fluctuation in internal energy was found to tend to zero in the thermodynamic limit. Both these results lead to an expectation that an extensive thermodynamics also will result from the microcanonical approach. This has been confirmed by detailed calculation for $a \geqslant -7/2$.

However, in the fifth case considered in Section 3, a was assumed less than $-5/2$, but condition (3.4) held and one particle was assumed appreciably heavier than the rest. In this special case, an unphysical entropy was found to result once again even after the small correction to the procedure of Ref. 4 was made. It should be noted that, in Ref. 3, the model considered for this case is similar to that considered in Ref. 4 and here, except that overall momentum conservation is imposed. The resulting entropy in this case, while still nonextensive, is superadditive. Hence, the model in Ref. 3 is acceptable physically and, incidentally, is seen to possess a negative heat capacity. This is consistent with the theorem[2] that a negative heat capacity implies a nonextensive entropy, provided superadditivity is satisfied.

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